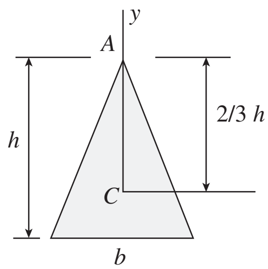
Polar Moments of Inertia

Problem 12.6-1 Determine the polar moment of inertia I_P of an isosceles triangle of base b and altitude h with respect to its apex (see Case 5, Appendix D)

Solution 12.6-1 **Polar moment of inertia**



POINT C (CENTROID) FROM CASE 5:

$$(I_P)_c = \frac{bh}{144}(4h^2 + 3b^2)$$

POINT A (APEX):

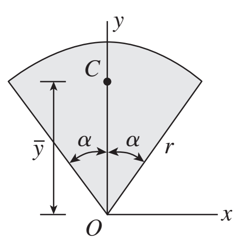
$$I_P = (I_P)_c + A\left(\frac{2h}{3}\right)^2$$

$$= \frac{bh}{144}(4h^2 + 3b^2) + \frac{bh}{2}\left(\frac{2h}{3}\right)^2$$

$$I_P = \frac{bh}{48}(b^2 + 12h^2) \longleftarrow$$

Problem 12.6-2 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a circular sector (see Case 13, Appendix D).

Solution 12.6-2 **Polar moment of inertia**



POINT O (ORIGIN) FROM CASE 13:

$$(I_P)_o = \frac{\alpha r^4}{2} \qquad (\alpha = \text{radians})$$

$$A = \alpha r^2$$

$$\bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

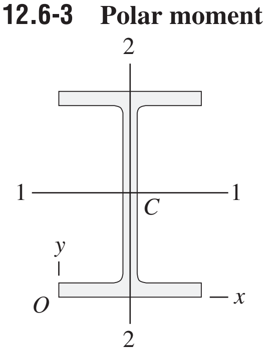
POINT C (CENTROID):

$$(I_P)_C = (I_P)_O - A\bar{y}^2 = \frac{\alpha r^4}{2} - \alpha r^2\left(\frac{2r \sin \alpha}{3\alpha}\right)^2$$

$$= \frac{r^4}{18\alpha}(9\alpha^2 - 8 \sin^2\alpha) \longleftarrow$$

Problem 12.6-3 Determine the polar moment of inertia I_P for a W 8 × 21 wide-flange section with respect to one of its outermost corners.

Solution 12.6-3 **Polar moment of inertia**



W 8 × 21 $\quad I_1 = 75.3 \text{ in.}^4 \quad I_2 = 9.77 \text{ in.}^4$

$A = 6.16 \text{ in.}^2$

Depth $d = 8.28$ in.

Width $b = 5.27$ in.

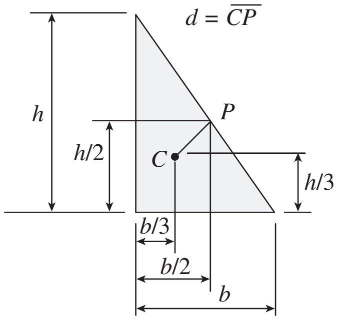
$$I_x = I_1 + A(d/2)^2 = 75.3 + 6.16(4.14)^2 = 180.9 \text{ in.}^4$$

$$I_y = I_2 + A(b/2)^2 = 9.77 + 6.16(2.635)^2 = 52.5 \text{ in.}^4$$

$$I_P = I_x + I_y = 233 \text{ in.}^4 \longleftarrow$$

Problem 12.6-4 Obtain a formula for the polar moment of inertia I_P with respect to the midpoint of the hypotenuse for a right triangle of base b and height h (see Case 6, Appendix D).

Solution 12.6-4 Polar moment of inertia



POINT C FROM CASE 6:

$$(I_P)_c = \frac{bh}{36}(h^2 + b^2)$$

POINT P:

$$I_P = (I_P)_c + Ad^2$$

$$A = \frac{bh}{2}$$

$$d^2 = \left(\frac{b}{2} - \frac{b}{3}\right)^2 + \left(\frac{h}{2} - \frac{h}{3}\right)^2$$

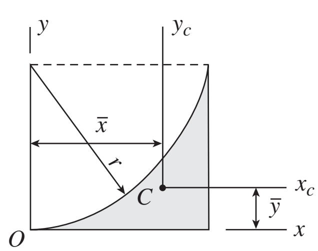
$$= \frac{b^2}{36} + \frac{h^2}{36} = \frac{b^2 + h^2}{36}$$

$$I_P = \frac{bh}{36}(h^2 + b^2) + \frac{bh}{2}\left(\frac{b^2 + h^2}{36}\right)$$

$$= \frac{bh}{24}(b^2 + h^2) \longleftarrow$$

Problem 12.6-5 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a quarter-circular spandrel (see Case 12, Appendix D).

Solution 12.6-5 Polar moment of inertia



POINT O FROM CASE 12:

$$I_x = \left(1 - \frac{5\pi}{16}\right)r^4$$

$$\bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)}$$

$$A = \left(1 - \frac{\pi}{4}\right)r^2$$

POINT C (CENTROID):

$$I_{x_c} = I_x - A\bar{y}^2 = \left(1 - \frac{5\pi}{16}\right)r^4 - \left(1 - \frac{\pi}{4}\right)(r^2)\left[\frac{(10 - 3\pi)r}{3(4 - \pi)}\right]^2$$

COLLECT TERMS AND SIMPLIFY:

$$I_{x_c} = \frac{r^4}{144}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right)$$

$$I_{y_c} = I_{x_c} \quad \text{(by symmetry)}$$

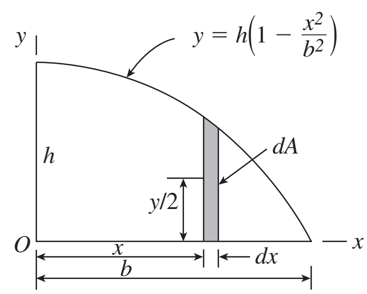
$$(I_P)_C = 2I_{x_c} = \frac{r^4}{72}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right) \longleftarrow$$

Products of Inertia

Problem 12.7-1 Using integration, determine the product of inertia I_{xy} for the parabolic semisegment shown in Fig. 12-5 (see also Case 17 in Appendix D).

Solution 12.7-1 Product of inertia

Product of inertia of element dA with respect to axes through its own centroid equals zero.



$$dA = y\,dx = h\left(1 - \frac{x^2}{b^2}\right)dx$$

dI_{xy} = product of inertia of element dA with respect to xy axes

$d_1 = x \qquad d_2 = y/2$

Parallel-axis theorem applied to element dA:

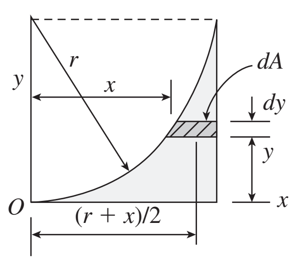
$$dI_{xy} = 0 + (dA)(d_1 d_2) = (y\,dx)(x)(y/2)$$

$$= \frac{h^2 x}{2}\left(1 - \frac{x^2}{b^2}\right)^2 dx$$

$$I_{xy} = \int dI_{xy} = \frac{h^2}{2}\int_0^b x\left(1 - \frac{x^2}{b^2}\right)^2 dx = \frac{b^2 h^2}{12} \quad \longleftarrow$$

Problem 12.7-2 Using integration, determine the product of inertia I_{xy} for the quarter-circular spandrel shown in Case 12, Appendix D.

Solution 12.7-2 Product of inertia



EQUATION OF CIRCLE:

$x^2 + (y - r)^2 = r^2$
or $r^2 - x^2 = (y - r)^2$

ELEMENT dA:

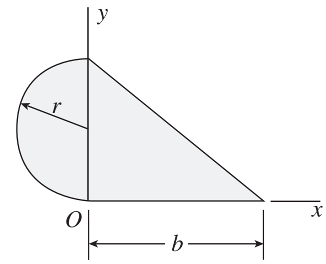
d_1 = distance to its centroid in x direction $= (r + x)/2$
d_2 = distance to its centroid in y direction $= y$
dA = area of element $= (r - x)\,dy$
Product of inertia of element dA with respect to axes through its own centroid equals zero.
Parallel-axis theorem applied to element dA:

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (r - x)(dy)\left(\frac{r + x}{2}\right)(y)$$

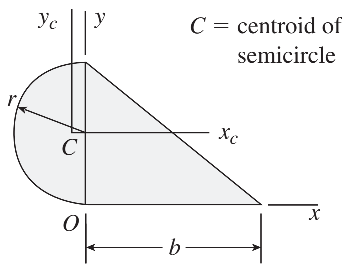
$$= \frac{1}{2}(r^2 - x^2)\,y\,dy = \frac{1}{2}(y - r)^2 y\,dy$$

$$I_{xy} = 1/2\int_0^r y(y - r)^2\,dy = \frac{r^4}{24} \quad \longleftarrow$$

Problem 12.7-3 Find the relationship between the radius r and the distance b for the composite area shown in the figure in order that the product of inertia I_{xy} will be zero.



Solution 12.7-3 Product of inertia



TRIANGLE (CASE 7):

$$I_{xy} = \frac{b^2h^2}{24} = \frac{b^2(2r)^2}{24} = \frac{b^2r^2}{6}$$

SEMICIRCLE (CASE 10):

$$I_{xy} = I_{x_cy_c} + Ad_1d_2$$

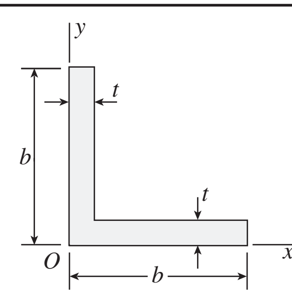
$$I_{x_cy_c} = 0 \qquad A = \frac{\pi r^2}{2} \qquad d_1 = r \qquad d_2 = -\frac{4r}{3\pi}$$

$$I_{xy} = 0 + \left(\frac{\pi r^2}{2}\right)(r)\left(-\frac{4r}{3\pi}\right) = -\frac{2r^4}{3}$$

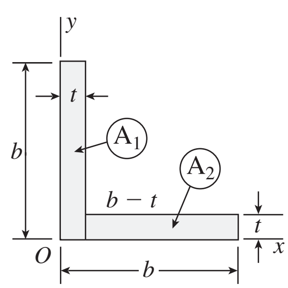
COMPOSITE AREA ($I_{xy} = 0$)

$$I_{xy} = \frac{b^2r^2}{6} - \frac{2r^4}{3} = 0 \qquad \therefore b = 2r \quad \longleftarrow$$

Problem 12.7-4 Obtain a formula for the product of inertia I_{xy} of the symmetrical L-shaped area shown in the figure.



Solution 12.7-4 Product of inertia



AREA 1:

$$(I_{xy})_1 = \frac{t^2b^2}{4}$$

AREA 2:

$$(I_{xy})_2 = I_{x_cy_c} + A_2d_1d_2$$

$$= 0 + (b - t)(t)(t/2)\left(\frac{b+t}{2}\right)$$

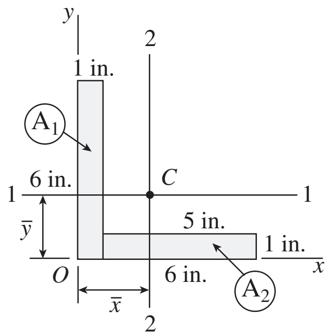
$$= \frac{t^2}{4}(b^2 - t^2)$$

COMPOSITE AREA:

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = \frac{t^2}{4}(2b^2 - t^2) \quad \longleftarrow$$

Problem 12.7-5 Calculate the product of inertia I_{12} with respect to the centroidal axes 1-1 and 2-2 for an ∟ 6 × 6 × 1 in. angle section (see Table E-4, Appendix E). (Disregard the cross-sectional areas of the fillet and rounded corners.)

Solution 12.7-5 Product of inertia



All dimensions in inches.

$A_1 = (6)(1) = 6.0\text{ in.}^2$

$A_2 = (5)(1) = 5.0\text{ in.}^2$

$A = A_1 + A_2 = 11.0\text{ in.}^2$

With respect to the x axis:

$$Q_1 = (6.0\text{ in.}^2)\left(\frac{6\text{ in.}}{2}\right) = 18.0\text{ in.}^3$$

$$Q_2 = (5.0\text{ in.}^2)\left(\frac{1.0\text{ in.}}{2}\right) = 2.5\text{ in.}^3$$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{20.5\text{ in.}^3}{11.0\text{ in.}^2} = 1.8636\text{ in.}$$

$\bar{x} = \bar{y} = 1.8636$ in.

Coordinates of centroid of area A_1 with respect to 1–2 axes:

$d_1 = -(\bar{x} - 0.5) = -1.3636$ in.

$d_2 = 3.0 - \bar{y} = 1.1364$ in.

Product of inertia of area A_1 with respect to 1-2 axes:

$I'_{12} = 0 + A_1 d_1 d_2$

$= (6.0\text{ in.}^2)(-1.3636\text{ in.})(1.1364\text{ in.}) = -9.2976\text{ in.}^4$

Coordinates of centroid of area A_2 with respect to 1–2 axes:

$d_1 = 3.5 - \bar{x} = 1.6364$ in.

$d_2 = -(\bar{y} - 0.5) = -1.3636$ in.

Product of inertia of area A_2 with respect to 1-2 axes:

$I''_{12} = 0 + A_2 d_1 d_2$

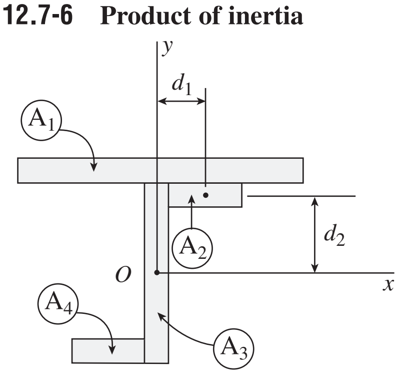
$= (5.0\text{ in.}^2)(1.6364\text{ in.})(-1.3636\text{ in.})$

$= -11.1573\text{ in.}^4$

ANGLE SECTION: $I_{12} = I'_{12} + I''_{12} = -20.5\text{ in.}^4$ ←

Problem 12.7-6 Calculate the product of inertia I_{xy} for the composite area shown in Prob. 12.3-6.

Solution 12.7-6 Product of inertia



All dimensions in millimeters

$A_1 = 360 \times 30$ mm $A_2 = 90 \times 30$ mm

$A_3 = 180 \times 30$ mm $A_4 = 90 \times 30$ mm

$d_1 = 60$ mm $d_2 = 75$ mm

AREA A_1: $(I_{xy})_1 = 0$ (By symmetry)

AREA A_2: $(I_{xy})_2 = 0 + A_2 d_1 d_2 = (90 \times 30)(60)(75)$
$= 12.15 \times 10^6\text{ mm}^4$

AREA A_3: $(I_{xy})_3 = 0$ (By symmetry)

AREA A_4: $(I_{xy})_4 = (I_{xy})_2 = 12.15 \times 10^6\text{ mm}^4$

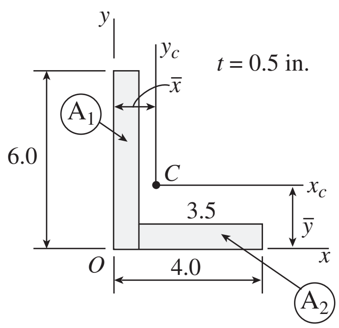
$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 + (I_{xy})_4$

$= (2)(12.15 \times 10^6\text{ mm}^4)$

$= 24.3 \times 10^6\text{ mm}^4$ ←

Problem 12.7-7 Determine the product of inertia $I_{x_c y_c}$ with respect to centroidal axes x_c and y_c parallel to the x and y axes, respectively, for the L-shaped area shown in Prob. 12.3-7.

Solution 12.7-7 Product of inertia



All dimensions in inches.

$A_1 = (6.0)(0.5) = 3.0 \text{ in.}^2$

$A_2 = (3.5)(0.5) = 1.75 \text{ in.}^2$

$A = A_1 + A_2 = 4.75 \text{ in.}^2$

With respect to the x axis:

$Q_1 = A_1\bar{y}_1 = (3.0 \text{ in.}^2)(3.0 \text{ in.}) = 9.0 \text{ in.}^3$

$Q_2 = A_2\bar{y}_2 = (1.75 \text{ in.}^2)(0.25 \text{ in.}) = 0.4375 \text{ in.}^3$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{9.4375 \text{ in.}^3}{4.75 \text{ in.}^2} = 1.9868 \text{ in.}$$

With respect to the y axis:

$Q_1 = A_1\bar{x}_1 = (3.0 \text{ in.}^2)(0.25 \text{ in.}) = 0.75 \text{ in.}^3$

$Q_2 = A_2\bar{x}_2 = (1.75 \text{ in.}^2)(2.25 \text{ in.}) = 3.9375 \text{ in.}^3$

$$\bar{x} = \frac{Q_1 + Q_2}{A} = \frac{4.6875 \text{ in.}^3}{4.75 \text{ in.}^2} = 0.98684 \text{ in.}$$

Product of inertia of area A_1 with respect to xy axes:

$$(I_{xy})_1 = (I_{xy})_{\text{centroid}} + A_1 d_1 d_2$$
$$= 0 + (3.0 \text{ in.}^2)(0.25 \text{ in.})(3.0 \text{ in.}) = 2.25 \text{ in.}^4$$

Product of inertia of area A_2 with respect to xy axes:

$$(I_{xy})_2 = (I_{xy})_{\text{centroid}} + A_2 d_1 d_2$$
$$= 0 + (1.75 \text{ in.}^2)(2.25 \text{ in.})(0.25 \text{ in.}) = 0.98438 \text{ in.}^4$$

ANGLE SECTION

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = 3.2344 \text{ in.}^4$$

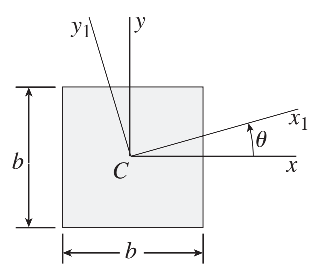
CENTROIDAL AXES

$$I_{x_c y_c} = I_{xy} - A\bar{x}\,\bar{y}$$
$$= 3.2344 \text{ in.}^4 - (4.75 \text{ in.}^2)(0.98684 \text{ in.})(1.9868 \text{ in.})$$
$$= -6.079 \text{ in.}^4 \quad \longleftarrow$$

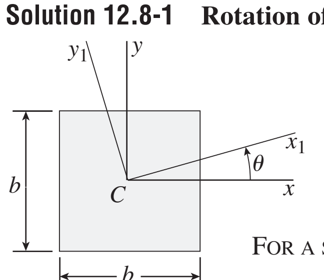
Rotation of Axes

The problems for Section 12.8 are to be solved by using the transformation equations for moments and products of inertia.

Problem 12.8-1 Determine the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1 y_1}$ for a square with sides b, as shown in the figure. (Note that the $x_1 y_1$ axes are centroidal axes rotated through an angle θ with respect to the xy axes.)



Solution 12.8-1 Rotation of axes



FOR A SQUARE:

$$I_x = I_y = -\frac{b^4}{12} \qquad I_{xy} = 0$$

EQ. (12-25):

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$
$$= \frac{I_x + I_y}{2} + 0 - 0 = \frac{b^4}{12} \quad \longleftarrow$$

EQ. (12-29):

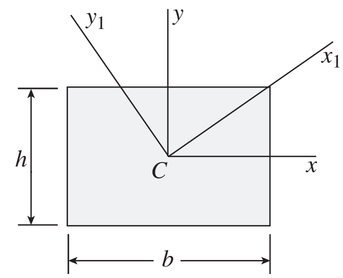
$$I_{x_1} + I_{y_1} = I_x + I_y \qquad \therefore I_{y_1} = \frac{b^4}{12} \quad \longleftarrow$$

EQ. (12-27):

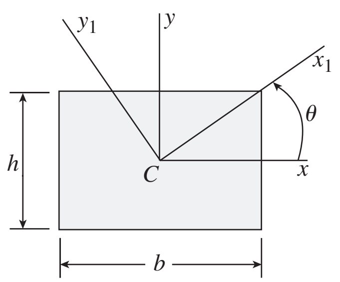
$$I_{x_1 y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 0 \quad \longleftarrow$$

Since θ may be any angle, we see that all moments of inertia are the same and the product of inertia is always zero (for axes through the centroid C).

Problem 12.8-2 Determine the moments and product of inertia with respect to the x_1y_1 axes for the rectangle shown in the figure. (Note that the x_1 axis is a diagonal of the rectangle.)



Solution 12.8-2 Rotation of axes (rectangle)



APPENDIX D, CASE 1:

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0$$

ANGLE OF ROTATION:

$$\cos\theta = \frac{b}{\sqrt{b^2 + h^2}} \qquad \sin\theta = \frac{h}{\sqrt{b^2 + h^2}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{b^2 - h^2}{b^2 + h^2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2bh}{b^2 + h^2}$$

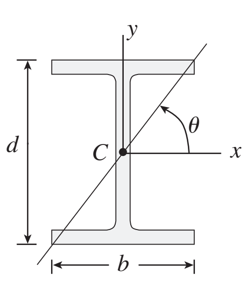
SUBSTITUTE INTO EQS. (12-25), (12-29), AND (12-27) AND SIMPLIFY:

$$I_{x_1} = \frac{b^3h^3}{6(b^2 + h^2)} \longleftarrow \qquad I_{y_1} = \frac{bh(b^4 + h^4)}{12(b^2 + h^2)} \longleftarrow$$

$$I_{x_1y_1} = \frac{b^2h^2(h^2 - b^2)}{12(b^2 + h^2)} \longleftarrow$$

Problem 12.8-3 Calculate the moment of inertia I_d for a W 12 × 50 wide-flange section with respect to a diagonal passing through the centroid and two outside corners of the flanges. (Use the dimensions and properties given in Table E-1.)

Solution 12.8-3 Rotation of axes



W 12 × 50 $\quad I_x = 394 \text{ in.}^4$
$I_y = 56.3 \text{ in.}^4 \quad I_{xy} = 0$
Depth $d = 12.19$ in.
Width $b = 8.080$ in.

$$\text{Tan}\,\theta = \frac{d}{b} = \frac{12.19}{8.080} = 1.509$$

$$\theta = 56.46^\circ \qquad 2\theta = 112.92^\circ$$

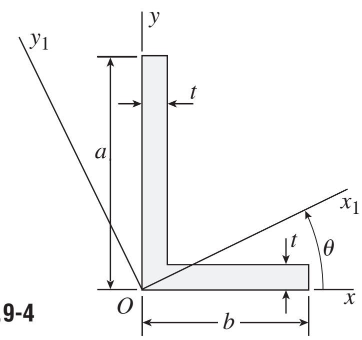
EQ. (12-25):

$$I_d = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \frac{394 + 56.3}{2} + \frac{394 - 56.3}{2}\cos(112.92^\circ) - 0$$

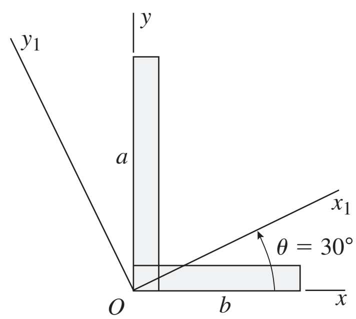
$$= 225 \text{ in.}^4 - 66 \text{ in.}^4 = 159 \text{ in.}^4 \longleftarrow$$

Problem 12.8-4 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the L-shaped area shown in the figure if $a = 150$ mm, $b = 100$ mm, $t = 15$ mm, and $\theta = 30°$.



Probs. 12.8-4 and 12.9-4

Solution 12.8-4 **Rotation of axes**



All dimensions in millimeters.

$a = 150$ mm $\quad b = 100$ mm

$t = 15$ mm $\quad \theta = 30°$

$$I_x = \frac{1}{3}ta^3 + \frac{1}{3}(b - t)t^3$$

$$= \frac{1}{3}(15)(150)^3 + \frac{1}{3}(85)(15)^3$$

$$= 16.971 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{3}(a - t)t^3 + \frac{1}{3}tb^3$$

$$= \frac{1}{3}(135)(15)^3 + \frac{1}{3}(15)(100)^3$$

$$= 5.152 \times 10^6 \text{ mm}^4$$

$$I_{xy} = \frac{1}{4}t^2a^2 + Ad_1d_2 \qquad A = (b-t)(t)$$

$$d_1 = t + \frac{b - t}{2} \qquad d_2 = \frac{t}{2}$$

$$I_{xy} = \frac{1}{4}(15)^2(150)^2 + (85)(15)(57.5)(7.5)$$

$$= 1.815 \times 10^6 \text{ mm}^4$$

Substitute into Eq. (12-25) with $\theta = 30°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= 12.44 \times 10^6 \text{ mm}^4 \longleftarrow$$

Substitute into Eq. (12-25) with $\theta = 120°$:

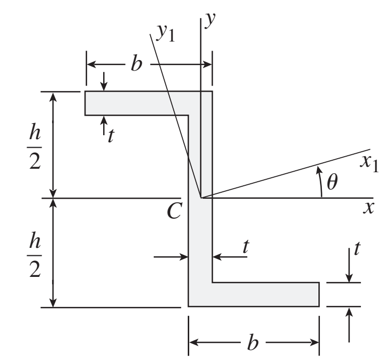
$$I_{y_1} = 9.68 \times 10^6 \text{ mm}^4 \longleftarrow$$

Substitute into Eq. (12-27) with $\theta = 30°$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

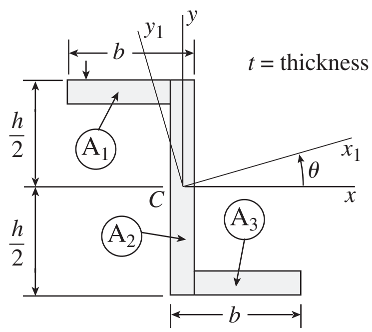
$$= 6.03 \times 10^6 \text{ mm}^4 \longleftarrow$$

Problem 12.8-5 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the Z-section shown in the figure if $b = 3$ in., $h = 4$ in., $t = 0.5$ in., and $\theta = 60°$.



Probs. 12.8-5, 12.8-6, 12.9-5 and 12.9-6

Solution 12.8-5 Rotation of axes



All dimensions in inches.

$b = 3.0$ in. $h = 4.0$ in. $t = 0.5$ in. $\theta = 60°$

MOMENT OF INERTIA I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$

$= 3.8542 \text{ in.}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 2.6667 \text{ in.}^4$

Area A_3: $I'''_x = I'_x = 3.8542 \text{ in.}^4$

$I_x = I'_x + I''_x + I'''_x = 10.3751 \text{ in.}^4$

MOMENT OF INERTIA I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$

$= 3.4635 \text{ in.}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.0417 \text{ in.}^4$

Area A_3: $I'''_y = I'_y = 3.4635 \text{ in.}^4$

$I_y = I'_y + I''_y + I'''_y = 6.9688 \text{ in.}^4$

PRODUCT OF INERTIA I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$

$= -\frac{1}{4}(bt)(b-t)(h-t) = -3.2813 \text{ in.}^4$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -6.5625 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 60°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$= 13.50 \text{ in.}^4$ ←

SUBSTITUTE into Eq. (12-25) with $\theta = 150°$:

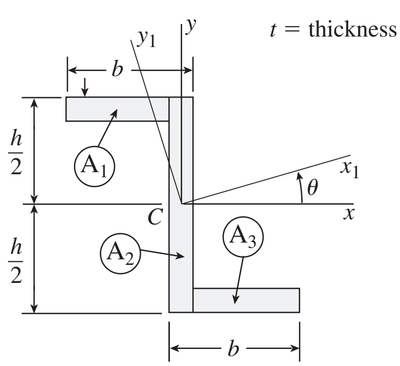
$I_{y_1} = 3.84 \text{ in.}^4$ ←

SUBSTITUTE into Eq. (12-27) with $\theta = 60°$:

$$I_{x_1 y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 4.76 \text{ in.}^4$$ ←

Problem 12.8-6 Solve the preceding problem if $b = 80$ mm, $h = 120$ mm, $t = 12$ mm, and $\theta = 30°$.

Solution 12.8-6 Rotation of axes



All dimensions in millimeters.

$b = 80$ mm $h = 120$ mm
$t = 12$ mm $\theta = 30°$

MOMENT OF INERTIA I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$

$= 2.3892 \times 10^6 \text{ mm}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 1.7280 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_x = I'_x = 2.3892 \times 10^6 \text{ mm}^4$

$I_x = I'_x + I''_x + I'''_x = 6.5065 \times 10^6 \text{ mm}^4$

(Continued)

MOMENT OF INERTIA I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$

$= 1.6200 \times 10^6 \text{ mm}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.01728 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_y = I'_y = 1.6200 \times 10^6 \text{ mm}^4$

$I_y = I'_y + I''_y + I'''_y = 3.2573 \times 10^6 \text{ mm}^4$

PRODUCT OF INERTIA I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$

$= -\frac{1}{4}(bt)(b-t)(h-t) = -1.7626 \times 10^6 \text{ mm}^4$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -3.5251 \times 10^6 \text{ mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 30°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$= 8.75 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-25) with $\theta = 120°$:

$I_{y_1} = 1.02 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-27) with $\theta = 30°$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$= -0.356 \times 10^6 \text{ mm}^4$ ←

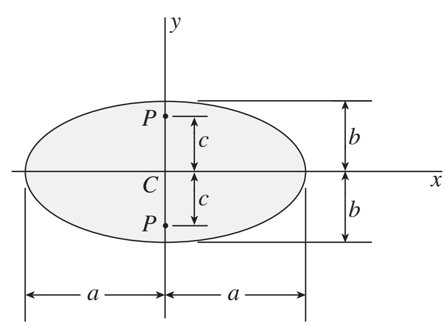
Principal Axes, Principal Points, and Principal Moments of Inertia

Problem 12.9-1 An ellipse with major axis of length $2a$ and minor axis of length $2b$ is shown in the figure.

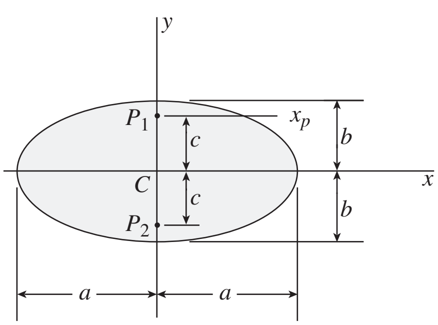
(a) Determine the distance c from the centroid C of the ellipse to the principal points P on the minor axis (y axis).

(b) For what ratio a/b do the principal points lie on the circumference of the ellipse?

(c) For what ratios do they lie inside the ellipse?



Solution 12.9-1 **Principal points of an ellipse**



(a) LOCATION OF PRINCIPAL POINTS

At a principal point, all moments of inertia are equal.

At point P_1: $I_{x_p} = I_y$ Eq. (1)

From Case 16: $I_y = \frac{\pi ba^3}{4}$

$I_x = \frac{\pi ab^3}{4}$ $A = \pi ab$

Parallal-axis theorem:

$$I_{x_p} = I_x + Ac^2 = \frac{\pi ab^3}{4} + \pi abc^2$$

Substitute into Eq. (1):

$$\frac{\pi ab^3}{4} + \pi abc^2 = \frac{\pi ba^3}{4}$$

Solve for c: $c = \frac{1}{2}\sqrt{a^2 - b^2}$ ←

(b) PRINCIPAL POINTS ON THE CIRCUMFERENCE

$\therefore c = b$ and $b = \dfrac{1}{2}\sqrt{a^2 - b^2}$

Solve for ratio $\dfrac{a}{b}$: $\quad \dfrac{a}{b} = \sqrt{5}$ ⟵

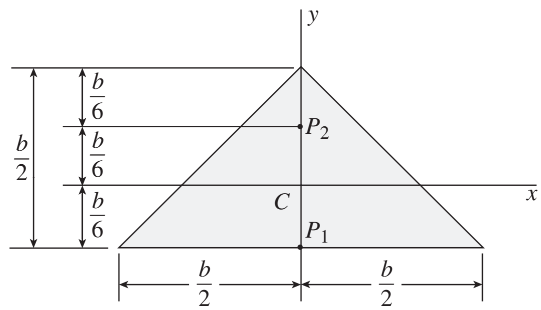
(c) PRINCIPAL POINTS INSIDE THE ELLIPSE

$\therefore 0 \le c < b \qquad$ For $c = 0$: $\quad a = b$ and $\dfrac{a}{b} = 1$

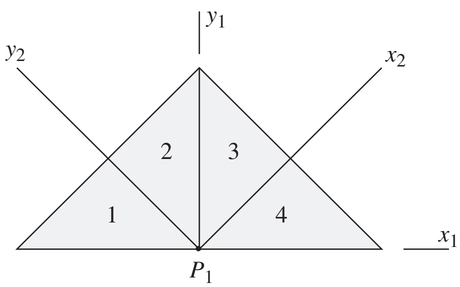
For $c = b$: $\quad \dfrac{a}{b} = \sqrt{5}$

$\therefore 1 \le \dfrac{a}{b} < \sqrt{5}$ ⟵

Problem 12.9-2 Demonstrate that the two points P_1 and P_2, located as shown in the figure, are the principal points of the isosceles right triangle.



Solution 12.9-2 Principal points of an isosceles right triangle



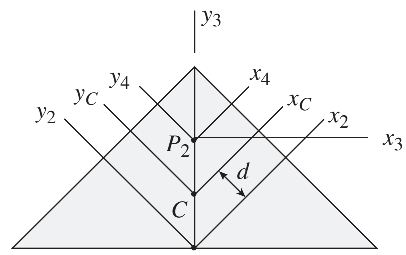
CONSIDER POINT P_1:

$I_{x_1 y_1} = 0$ because y_1 is an axis of symmetry.

$I_{x_2 y_2} = 0$ because areas 1 and 2 are symmetrical about the y_2 axis and areas 3 and 4 are symmetrical about the x_2 axis.

Two different sets of principal axes exist at point P_1.

$\therefore P_1$ is a principal point ⟵



CONSIDER POINT P_2:

$I_{x_3 y_3} = 0$ because y_3 is an axis of symmetry.

$I_{x_2 y_2} = 0$ (see above).

Parallel-axis theorem:

$$I_{x_2 y_2} = I_{x_c y_c} + A d_1 d_2 \qquad A = \frac{b^2}{4} \qquad d = d_1 = d_2 = \frac{b}{6\sqrt{2}}$$

$$I_{x_c y_c} = -\left(\frac{b^2}{4}\right)\left(\frac{b}{6\sqrt{2}}\right)^2 = -\frac{b^4}{288}$$

Parallel-axis theorem:

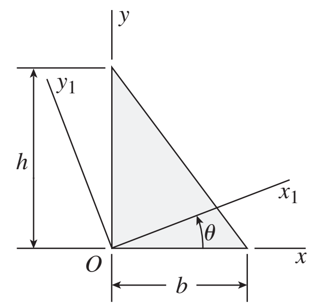
$$I_{x_4 y_4} = I_{x_c y_c} + A d_1 d_2 \qquad d_1 = d_2 = -\frac{b}{6\sqrt{2}}$$

$$I_{x_4 y_4} = -\frac{b^4}{288} + \frac{b^2}{4}\left(-\frac{b}{6\sqrt{2}}\right)^2 = 0$$

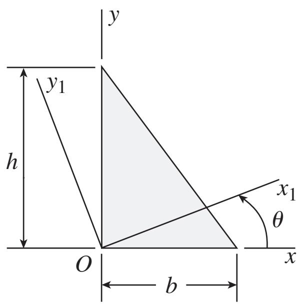
Two different sets of principal axes ($x_3 y_3$ and $x_4 y_4$) exist at point P_2.

$\therefore P_2$ is a principal point ⟵

Problem 12.9-3 Determine the angles θ_{p1} and θ_{p2} defining the orientations of the principal axes through the origin O for the right triangle shown in the figure if $b = 6$ in. and $h = 8$ in. Also, calculate the corresponding principal moments of inertia I_1 and I_2.



Solution 12.9-3 Principal axes



RIGHT TRIANGLE

$b = 6.0$ in. $\quad h = 8.0$ in.

CASE 7:

$$I_x = \frac{bh^3}{12} = 256 \text{ in.}^4$$

$$I_y = \frac{hb^3}{12} = 144 \text{ in.}^4$$

$$I_{xy} = \frac{b^2h^2}{24} = 96 \text{ in.}^4$$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -1.71429$

$2\theta_p = -59.744°$ and $120.256°$

$\theta_p = -29.872°$ and $60.128°$

SUBSTITUTE into Eq. (12-25) with $\theta = -29.872°$:

$I_{x_1} = 311.1 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 60.128°$:

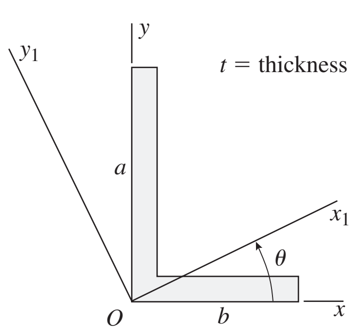
$I_{x_1} = 88.9 \text{ in.}^4$

THEREFORE, $\left.\begin{array}{ll} I_1 = 311.1 \text{ in.}^4 & \theta_{p_1} = -29.87° \\ I_2 = 88.9 \text{ in.}^4 & \theta_{p_2} = 60.13° \end{array}\right\} \leftarrow$

NOTE: The principal moments of inertia can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-4 Determine the angles θ_{p1} and θ_{p2} defining the orientations of the principal axes through the origin O and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area described in Prob. 12.8-4 ($a = 150$ mm, $b = 100$ mm, and $t = 15$ mm).

Solution 12.9-4 Principal axes



ANGLE SECTION

$a = 150$ mm $\quad b = 100$ mm $\quad t = 15$ mm

FROM PROB. 12.8-4:

$I_x = 16.971 \times 10^6 \text{ mm}^4$

$I_y = 5.152 \times 10^6 \text{ mm}^4 \quad I_{xy} = 1.815 \times 10^6 \text{ mm}^4$

EQ. (12-30): $\quad \tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.3071$

$2\theta_p = -17.07°$ and $162.93°$

$\theta_p = -8.54°$ and $81.46°$

SUBSTITUTE into Eq. (12-25) with $\theta = -8.54°$:

$I_{x_1} = 17.24 \times 10^6\,\text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 81.46°$:

$I_{x_1} = 4.88 \times 10^6\,\text{mm}^4$

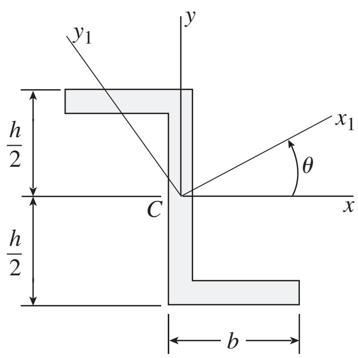
THEREFORE,

$I_1 = 17.24 \times 10^6\ \text{mm}^4 \quad \theta_{p_1} = -8.54°$
$I_2 = 4.88 \times 10^6\ \text{mm}^4 \quad \theta_{p_2} = 81.46°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-5 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C and the corresponding principal centroidal moments of inertia I_1 and I_2 for the Z-section described in Prob. 12.8-5 (b = 3 in., h = 4 in., and t = 0.5 in.).

Solution 12.9-5 Principal axes



Z-SECTION

t = thickness = 0.5 in.
b = 3.0 in $\quad h$ = 4.0 in

FROM PROB. 12.8-5:

$I_x = 10.3751\ \text{in.}^4 \quad I_y = 6.9688\ \text{in.}^4$
$I_{xy} = -6.5625\ \text{in.}^4$

EQ. (12-30): $\quad \tan 2\theta_p = -\dfrac{2\,I_{xy}}{I_x - I_y} = 3.8532$

$2\theta_p = 75.451° \quad \text{and} \quad 255.451°$

$\theta_p = 37.726° \quad \text{and} \quad 127.726°$

SUBSTITUTE into Eq. (12-25) with $\theta = 37.726°$:

$I_{x_1} = 15.452\,\text{in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 127.726°$:

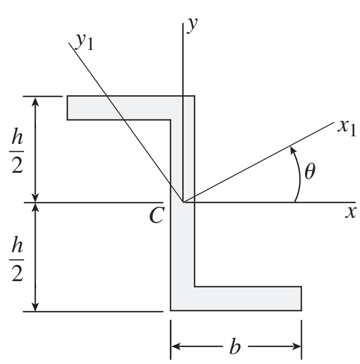
$I_{x_1} = 1.892\,\text{in.}^4$

THEREFORE, $\left.\begin{array}{ll} I_1 = 15.45\ \text{in.}^4 & \theta_{p_1} = 37.73° \\ I_2 = 1.89\ \text{in.}^4 & \theta_{p_2} = 127.73° \end{array}\right\} \longleftarrow$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-6 Solve the preceding problem for the Z-section described in Prob. 12.8-6 (b = 80 mm, h = 120 mm, and t = 12 mm).

Solution 12.9-6 Principal axes



Z-SECTION

t = thickness
= 12 mm
b = 80 mm
h = 120 mm

FROM PROB. 12.8-6:

$I_x = 6.5065 \times 10^6\ \text{mm}^4 \quad I_y = 3.2573 \times 10^6\ \text{mm}^4$
$I_{xy} = -3.5251 \times 10^6\ \text{mm}^4$

(Continued)

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = 2.1698$

$2\theta_p = 65.257°$ and $245.257°$
$\theta_p = 32.628°$ and $122.628°$

SUBSTITUTE into EQ. (12-25) with $\theta = 32.628°$:

$I_{x_1} = 8.763 \times 10^6 \text{ mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 122.628°$:

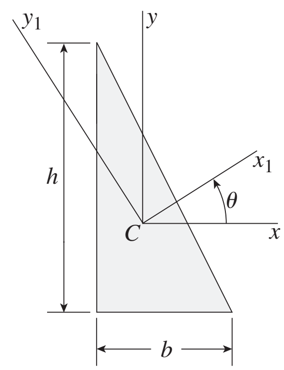
$I_{x_1} = 1.000 \times 10^6 \text{ mm}^4$

THEREFORE,

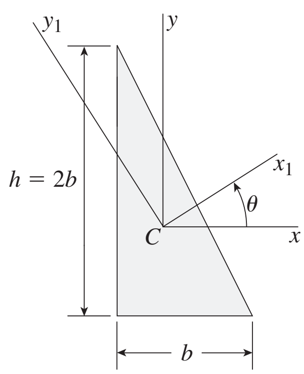
$$\left.\begin{array}{ll} I_1 = 8.76 \times 10^6 \text{ mm}^4 & \theta_{p_1} = 32.63° \\ I_2 = 1.00 \times 10^6 \text{ mm}^4 & \theta_{p_2} = 122.63° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-7 Determine the angles θ_{p1} and θ_{p2} defining the orientations of the principal axes through the centroid C for the right triangle shown in the figure if $h = 2b$. Also, determine the corresponding principal centroidal moments of inertia I_1 and I_2.



Solution 12.9-7 Principal axes



RIGHT TRIANGLE

$h = 2b$

CASE 6

$$I_x = \frac{bh^3}{36} = \frac{2b^4}{9}$$

$$I_y = \frac{hb^3}{36} = \frac{b^4}{18}$$

$$I_{xy} = -\frac{b^2h^2}{72} = -\frac{b^4}{18}$$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = \dfrac{2}{3}$

$2\theta_p = 33.6901°$ and $213.6901°$
$\theta_p = 16.8450°$ and $106.8450°$

SUBSTITUTE into Eq. (12-25) with $\theta = 16.8450°$:

$I_{x_1} = 0.23904\, b^4$

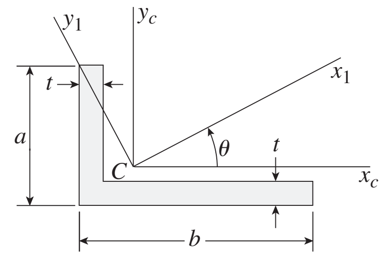
SUBSTITUTE into Eq. (12-25) with $\theta = 106.8450°$:

$I_{x_1} = 0.03873\, b^4$

THEREFORE, $$\left.\begin{array}{ll} I_1 = 0.2390\, b^4 & \theta_{p_1} = 16.85° \\ I_2 = 0.0387\, b^4 & \theta_{p_2} = 106.85° \end{array}\right\} \longleftarrow$$

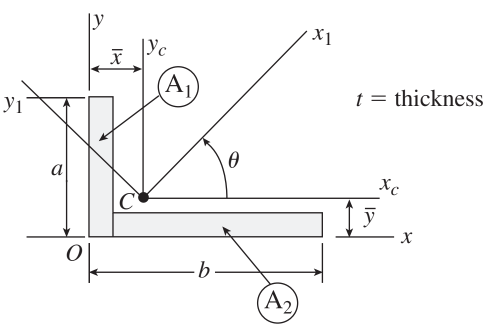
NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-8 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal centroidal axes and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area shown in the figure if $a = 80$ mm, $b = 150$ mm, and $t = 16$ mm.



Probs. 12.9-8 and 12.9-9

Solution 12.9-8 **Principal axes (angle section)**



$a = 80$ mm $\quad b = 150$ mm $\quad t = 16$ mm

$A_1 = at = 1280 \text{ mm}^2$

$A_2 = (b - t)(t) = 2144 \text{ mm}^2$

$A = A_1 + A_2 = t\,(a + b - t) = 3424 \text{ mm}^2$

LOCATION OF CENTROID C

$$Q_x = \sum A_i \bar{y}_i = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right)$$

$$= 68{,}352 \text{ mm}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{68{,}352 \text{ mm}^3}{3{,}424 \text{ mm}^2} = 19.9626 \text{ mm}$$

$$Q_y = \sum A_i \bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right)$$

$$= 188{,}192 \text{ mm}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{188{,}192 \text{ mm}^3}{3{,}424 \text{ mm}^2} = 54.9626 \text{ mm}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$

$$= \frac{1}{12}(16)(80)^3 + (1280)(40)^2 + \frac{1}{12}(134)(16)^3 + (2144)(8)^2$$

$$= 2.91362 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3) + A_2\left(\frac{b + t}{2}\right)^2$$

$$= \frac{1}{12}(80)(16)^3 + (1280)(8)^2 + \frac{1}{12}(16)(134)^3 + (2144)\left(\frac{166}{2}\right)^2$$

$$= 18.08738 \times 10^6 \text{ mm}^4$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 2.91362 \times 10^6 - (3424)(19.9626)^2$$
$$= 1.54914 \times 10^6 \text{ mm}^4$$

$$I_{y_c} = I_y - A\bar{x}^2 = 18.08738 \times 10^6 - (3424)(54.9626)^2$$
$$= 7.74386 \times 10^6 \text{ mm}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $\quad I_{xy} = I_{\text{centroid}} + A\,d_1 d_2$

Area A_1: $I'_{x_c y_c} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{e}{2} - \bar{y}\right]$

$$= (1280)(8 - 54.9626)(40 - 19.9626)$$
$$= -1.20449 \times 10^6 \text{ mm}^4$$

Area A_2: $I''_{x_c y_c} = 0 + A_2\left[\frac{b + t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$

$$= (2144)(83 - 54.9626)(8 - 19.9626)$$
$$= -0.71910 \times 10^6 \text{ mm}^4$$

$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -1.92359 \times 10^6 \text{ mm}^4$$

SUMMARY

$I_{x_c} = 1.54914 \times 10^6 \text{ mm}^4 \qquad I_{y_c} = 7.74386 \times 10^6 \text{ mm}^4$

$I_{x_c y_c} = -1.92359 \times 10^6 \text{ mm}^4$

(Continued)

PRINCIPAL AXES

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.621041$

$2\theta_p = -31.8420°$ and $148.1580°$

$\theta_p = -15.9210°$ and $74.0790°$

SUBSTITUTE into Eq. (12-25) with $\theta = -15.9210°$

$I_{x_1} = 1.0004 \times 10^6\,\text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 74.0790°$

$I_{x_1} = 8.2926 \times 10^6\,\text{mm}^4$

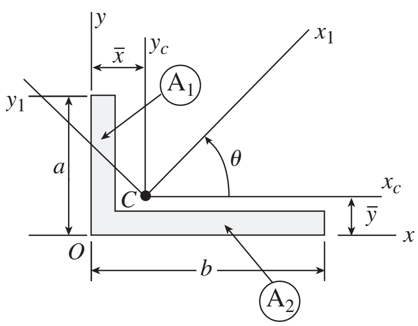
THEREFORE,

$$\left.\begin{array}{ll} I_1 = 8.29 \times 10^6\ \text{mm}^4 & \theta_{p_1} = 74.08° \\ I_2 = 1.00 \times 10^6\ \text{mm}^4 & \theta_{p_2} = -15.92° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-9 Solve the preceding problem if $a = 3$ in., $b = 6$ in., and $t = 5/8$ in.

Solution 12.9-9 Principal axes (angle section)



$a = 3.0$ in.
$b = 6.0$ in.
$t = 5/8$ in.
$A_1 = at = 1.875\ \text{in.}^2$
$A_2 = (b - t)(t) = 3.35938\ \text{in.}^2$
$A = A_1 + A_2 = t(a + b - t) = 5.23438\ \text{in.}^2$

LOCATION OF CENTROID C

$$Q_x = \sum A_i\bar{y}_i = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right) = 3.86230\ \text{in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3.86230\ \text{in.}^3}{5.23438\ \text{in.}^2} = 0.73787\ \text{in.}$$

$$Q_y = \sum A_i\bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right) = 11.71387\ \text{in.}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{11.71387\ \text{in.}^3}{5.23438\ \text{in.}^2} = 2.23787\ \text{in.}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$
$$= \frac{1}{12}\left(\frac{5}{8}\right)(3.0)^3 + (1.875)(1.5)^2 + \frac{1}{12}(5.375)\left(\frac{5}{8}\right)^3 + (3.35938)\left(\frac{5}{16}\right)^2$$
$$= 6.06242\ \text{in.}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3) + A_2\left(\frac{b + t}{2}\right)^2$$
$$= \frac{1}{12}(3.0)\left(\frac{5}{8}\right)^3 + (1.875)\left(\frac{5}{16}\right)^2 + \frac{1}{12}\left(\frac{5}{8}\right)(5.375)^3 + (3.35938)\left(\frac{6.625}{2}\right)^2$$
$$= 45.1933\ \text{in.}^4$$

MOMENTS OF INERTIA (x_cy_c AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 6.06242 - (5.23438)(0.73787)^2 = 3.21255\ \text{in.}^4$$

$$I_{y_c} = I_y - A\bar{x}^2 = 45.1933 - (5.23438)(2.23787)^2 = 18.97923\ \text{in.}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $I_{xy} = I_{\text{centroid}} + A\, d_1 d_2$

$$\text{Area } A_1\text{: } I'_{x_C y_C} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{a}{2} - \bar{y}\right]$$
$$= (1.875)(-1.92537)(0.76213)$$
$$= -2.75134 \text{ in.}^4$$

$$\text{Area } A_2\text{: } I''_{x_C y_C} = 0 + A_2\left[\frac{b+t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$$
$$= (3.35938)(1.07463)(-0.42537)$$
$$= -1.53562 \text{ in.}^4$$

$$I_{x_C y_C} = I'_{x_C y_C} + I''_{x_C y_C} = -4.28696 \text{ in.}^4$$

SUMMARY

$I_{x_C} = 3.21255 \text{ in.}^4 \qquad I_{y_C} = 18.97923 \text{ in.}^4$

$I_{x_C y_C} = -4.28696 \text{ in.}^4$

PRINCIPAL AXES

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.54380$

$2\theta_p = -28.5374°$ and $151.4626°$
$\theta_p = -14.2687°$ and $75.7313°$

SUBSTITUTE into Eq. (12-25) with $\theta = -14.2687°$

$I_{x_1} = 2.1223 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 75.7313°$

$I_{x_1} = 20.0695 \text{ in.}^4$

THEREFORE,

$$\left.\begin{array}{ll} I_1 = 20.07 \text{ in.}^4 & \theta_{p_1} = 75.73° \\ I_2 = 2.12 \text{ in.}^4 & \theta_{p_2} = -14.27° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).